Divisors and line bundles

X: smooth connected variety over $k = \bar{k}$

$\Rightarrow$ X integral with generic pt η

Weil divisors

Def A prime divisor $Y \subseteq X$ is a closed integral subscheme w/ $\operatorname{codim}_X Y = 1$

A (Weil) divisor $D = \sum n_i Y_i \in \operatorname{Div}(X) := \bigoplus_{\substack{Y \subseteq X \\ \text{prime Div}}} \mathbb{Z} Y$

effective if all $n_i \geq 0$

$Y \subseteq X$: prime divisor

X smooth $\Rightarrow$ $\mathcal{O}_{X,Y}$ regular local ring of dim 1

i.e. a DVR w/ fraction field $K(X)$

$\leadsto$ valuation $\operatorname{ord}_Y : K(X)^* \longrightarrow \mathbb{Z}$

$f = ut^n \longmapsto n$ for $u \in \mathcal{O}_{X,Y}^\times$

$t \in \mathcal{O}_{X,Y}$ uniformizer a.k.a. local parameter

Def divisor of zeros and poles homomorphism

$$\operatorname{div} : K(X)^\times \longrightarrow \operatorname{Div}(X)$$
$$f \longmapsto \sum_{\substack{Y \subseteq X \\ \text{prime}}} \operatorname{ord}_Y(f)\, Y$$

Divisors of the form $\operatorname{div}(f)$ are called principal.

D, D' are linearly equivalent if $\exists f$ s.t. $D - D' = \operatorname{div}(f)$

The divisor class group is

$$\operatorname{Cl}(X) = \operatorname{Div}(X) / \operatorname{im}(\operatorname{div})$$

Prop A noetherian domain. TFAE:

(1). A is a UFD

(2) $\operatorname{Spec}(A)$ is normal and $\operatorname{Cl}(\operatorname{Spec}(A)) = 0$

(3) every prime ideal of ht $=1$ is principal

Fact if R regular local ring, then R is a UFD

⇝ for us $Cl(X)$ is a global/geometric object (in general, it also contains arithmetic information)

Cartier divisors

$\mathcal{K}_X$: constant sheaf on X valued in $K(X)$ — $\mathcal{O}_X$-algebra

$\cup\!\!\mid$

$\mathcal{K}_X^*$: ——— " ——— $K^\times(X)$

sheaf of abelian groups

consider s.e.s. (of abelian groups written multiplicatively)

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 1$$

Def A cartier divisor $\in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^\times)$

$$\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^\times) \cong \left\{ (U_i, f_i) \;\middle|\; \begin{array}{l} X = \bigcup U_i \\ f_i \in K(X)^\times \\ \text{s.t. } f_i/f_j \in \mathcal{O}_X(U_{ij})^\times \end{array} \right\} \Big/ \sim$$

Cartier divisors in the image of

$$K(X)^\times = \Gamma(X, \mathcal{K}_X^\times) \longrightarrow \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$$

are called principal

$\forall Y$ prime $\operatorname{ord}_Y|_{\mathcal{O}_X^\times} = 0$

$$\Rightarrow \begin{tikzcd} K(X)^\times \arrow[r] \arrow[dr, "\operatorname{ord}_Y"'] & \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \arrow[d, "\operatorname{ord}_Y"] \\ & \mathbb{Z} \end{tikzcd}$$

$$⇝ \begin{tikzcd} K(X)^\times \arrow[r] \arrow[dr] & \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \arrow[d, "\operatorname{div}"] \\ & \operatorname{Div}(X) \end{tikzcd}$$

Div(X)

Prop: X : smooth connected variety

$\mathrm{div} : \Gamma(X, \mathcal{K}^*_X / \mathcal{O}^*_X) \to \mathrm{Div}(X)$

is an isomorphism.

Proof $\forall x \in X$ $\mathcal{O}_{X,x}$ regular local domain $\Rightarrow$ UFD

$\forall Y \subseteq X$ prime $\quad Y|_{\mathrm{Spec}(\mathcal{O}_{X,x})} \longleftrightarrow$ cut out by a height 1 prime ideal

$$\mathfrak{p}_x = (f^Y_x)$$

$\Rightarrow \exists\ X = \bigcup_i U^Y_i$ open cover and $f^Y_i \in \mathcal{O}_X(U^Y_i)$ s.t.

$$Y|_{U^Y_i} = V(f^Y_i)$$

the map

$$\mathrm{Div}(X) \longrightarrow \Gamma(X, \mathcal{K}^*_X / \mathcal{O}^*_X)$$
$$Y \longmapsto (U^Y_i, f^Y_i)$$

yields an inverse to div. $\square$

Exercise $\mathrm{div}(U_i, f_i)$ is effective iff $f_i \in \mathcal{O}^X_{U_i}$

Line bundles

Def A line bundle (a.k.a invertible sheaf) $\mathcal{L}$ on X is a sheaf of $\mathcal{O}_X$-modules which is Zariski locally isomorphic to $\mathcal{O}_X$. i.e. $\exists$ open cover $X = \bigcup_i U_i$ and isomorphisms $\mathcal{O}_{U_i} \cong \mathcal{L}|_{U_i}$

$$\mathcal{O}_X(-) : \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \longrightarrow \left\{ (\mathcal{L}, s) \,\middle|\, \begin{array}{l} \mathcal{L} \text{ line bundle} \\ s \in \mathcal{L}_\eta \end{array} \right\} / \cong$$

$$D = (U_i, f_i) \mapsto \left(\mathcal{O}_X(D)(U_i) = f_i^{-1} \mathcal{O}_X \subseteq \mathcal{K}_X,\ s_D = 1 \right)$$

$$= \{ f \in K(X) \mid \operatorname{div}(f)|_U + D \geq 0 \} \cup \{0\},$$

Prop $(\mathcal{O}_X(-), s_D)$ is an isomorphism and induces an isomorphism

$$\begin{array}{ccc} \operatorname{CaCl}(X) & \longrightarrow & \operatorname{Pic}(X) \\ \wr\wr & & \\ \operatorname{Cl}(X) & & \end{array}$$

Proof Inverse construction:

$$\operatorname{div}(\mathcal{L}, s) := (U_i, \varphi_i(s))$$

for $X = \bigcup U_i$, $\varphi_i : \mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ a trivialisation.

$\square$

Example $D = [0] - 2[\infty] \in \mathrm{Div}(\mathbb{P}^1)$ $\quad \mathbb{P}^1 = \mathrm{Spec}(k[t]) \cup \mathrm{Spec}(k[s])$

what is the corresponding Cartier divisor

$$[0]|_{\mathrm{Spec}(k[t])} = \mathrm{Spec}(k[t]/(t))$$

$$[\infty]|_{\mathrm{Spec}(k[s])} = \mathrm{Spec}(k[s]/(s))$$

$$[0] \leftrightarrow (U_0, t) \quad (U_1, 1)$$

$$[\infty] \leftrightarrow (U_0, 1) \quad (U_1, s)$$

$$\Rightarrow \quad D = (U_0, t), (U_1, s^{-2})$$

$$\mathcal{O}_X(D)|_{U_0} = t\, k[t] \quad , \quad \mathcal{O}_X(D)|_{U_1} = s^{-2} k[s]$$

$$\mathcal{O}_X(D)|_{U_{01}} = t\, k[t^{\pm 1}] \overset{\simeq}{\longleftrightarrow} \mathcal{O}_X(D)|_{U_{10}} = s^{-2} k[s^{\pm 1}]$$

$$f(t) \longmapsto \frac{s^{-2}}{t} f(s^{-1}) = s^{-1} f(s^{-1})$$

$$t^{-1} g(t^{-1}) = \frac{t}{s^{-2}} g(t^{-1}) \longleftarrow g(s)$$

$$\begin{array}{ccccc}
\{(\mathcal{L}, s)\}/\simeq & \cong & \Gamma(X, \mathcal{K}^*_X/\mathcal{O}^*_X) & \cong & \mathrm{Div}(X) \\
\downarrow & /// & \downarrow & /// & \downarrow \\
\mathrm{Pic}(X) & \simeq & \mathrm{CaCl}(X) & \cong & \mathrm{Cl}(X)
\end{array}$$

$\Rightarrow$ set of divisors linearly equivalent to D is $\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\} / k^*$

[Hartshorne, II.7]

Linear systems

Def The complete linear system of a divisor D is

$$|D| = \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\} / k^* \cong \mathbb{P}(\Gamma(X, \mathcal{O}_X(D)))$$

the set of divisors which are linearly equivalent to D

A linear system of D is a linear subspace $\delta \subseteq |D|$

A point $p \in X$ is a base point for $\delta \subseteq |D|$ if $p \in \operatorname{supp}(D')$ $\forall D' \in \delta$

Maps to $\mathbb{P}^n$

$\mathcal{O}(1)$ is the line bundle on $\mathbb{P}^n$ w/ global sections $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = k[x_0, \dots, x_n]_1$

homogeneous of deg 1
linear forms

For all $X \xrightarrow{\pi} \mathbb{P}^n$ $\pi^* \mathcal{O}(1)$ is a line bundle which is globally generated by $\pi^* x_i$

Conversely:
Let $(\mathcal{L}, s_0, \dots, s_n)$ line bundle w/ globally gen. sections on X

$$\rightsquigarrow \quad X \xrightarrow{\pi} \mathbb{P}^n$$
$$p \longmapsto [s_0(p), \dots, s_n(p)]$$

s.t. $\pi^* \mathcal{O}(1) = \mathcal{L}$ and $\pi^* x_i = s_i$

Thm

$$\{X \to \mathbb{P}^n\} = \left\{ (\mathcal{L}, s_0, \dots, s_n) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ line bundle on } X \\ s_0, \dots, s_n \text{ globally generating sections} \end{array} \right\} / \cong$$

$$= \left\{ \delta, s_0, \ldots, s_n \;\middle|\; \begin{array}{l} \delta \text{ linear system of dim } n \\ s_i \text{ global generators} \end{array} \right\}$$